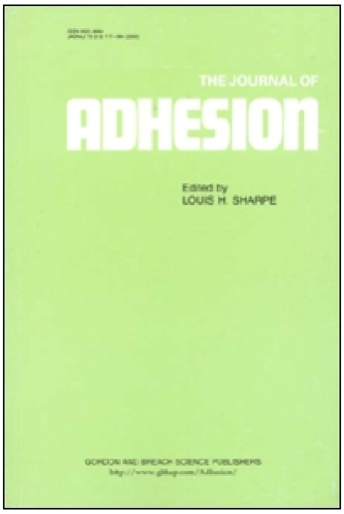

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An Analytical Model of the Mechanical Behaviour of Elastic Adhesively Bonded Joints

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Contact and adhesion between two elastic solids bonded by a thin elastic adhesive are considered. The peculiar case of a rigid flat cylindrical punch being adhered to an elastic layer is considered. In order to evaluate the influence of adhesive thickness on the mechanical behaviour of the bonded solids, an approximate solution of the contact problem is derived, using the Ritz method. Comparison of the analytical solution with FE calculations shows a good convergence towards the asymptotic value when the ratio of the punch radius to the layer thickness is greater than 4.

Numerical simulation reveals a strong effect of the adhesive compressibility, for a thin adhesive. This effect is assessed quantitatively from a double asymptotic expansion of the analytical solution. Lastly, the rupture stress is evaluated as a function of the adhesive thickness, from the knowledge of the solution of the contact problem; the asymptotic value for a very thin adhesive is determined independently.

KEY WORDS: Adhesion of a punch; analytical model; asymptotic expansion; variational method; fracture mechanics; tensile failure

1. INTRODUCTION

This paper addresses the problem of modelling the mechanical behaviour of two solids (the adherends) bonded together by a thin adhesive elastic layer. Industrial applications of this problem are more and more involved in the recent years due to the progress in bonding techniques which enable the use of adhesive bonding in high load situations with an increased confidence.[1] Moreover, several mechanical tests of adhesion (*e.g.* the peel test, the double cantilever beam, pull-out or fragmentation of single fiber composites) fall into this category of problems, which appear therefore equally pertinent from a scientific point of view. A further related problem of outstanding importance concerns the modelling of elastic multi-structures, *i.e.* elastic bodies made of different substructures of possibly different dimensions (three-dimensional substructures, plates, shells, rods) and properties, which very commonly occur: folded plates, H-shaped beams, plates clamped in three-dimensional foundations, plates or shells with stiffeners.

Evaluation of stresses transmitted from the adherends through the adhesive in the general case is a quite complicated task, since it would imply consideration of the

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adhesive as a three-dimensional body undergoing large displacement or/and large strains (this is typically the case in the peel test, which involves large displacements and rotations near the peel front); in addition, the adhesive has generally a viscous behaviour, which is both time and temperature dependent. Therefore, a simplified modelling of the adhesive behaviour is needed, in order to make the problem tractable from an analytical point of view.

From the pioneering work of Goland *et al.*[2], there have been numerous stress analysis of adhesive joints, where assumptions concerning the displacement and/or stress fields in the adhesive have been made. Following this work, models where the governing equations are simplified in order to eliminate the dependence of the through-the-thickness coordinate were obtained, thus considering the adhesive as a material surface. This kind of description is equivalent to evaluating averaged stresses through the thickness of the adhesive making the comparison with experimental measurements (averaged stresses are those directly evaluated experimentally) feasible. From this description, both analytical models and special finite element methods were given by Reddy[3] and Carpenter.[4]

The now classical plate theories fall into the same category of approaches, which simplify the original three-dimensional problem into a problem having a lower dimensionality: these are the Von-Karman equations for the two-dimensional, non-linear plate theory, and the Love-Kirchhoff theory in the linear case, see Ciarlet.[5]

2. REVIEW OF THE ASYMPTOTIC METHOD

2.1 Generalities

In a second type of method, called the asymptotic method, one tries to construct the solution of the three-dimensional problem as a series development of the unknowns in terms of a small non-dimensional parameter, ε (for instance, the ratio of the plate thickness to a characteristic macroscopic length); the first term in the series represents the limit as $\varepsilon \rightarrow 0$, which is an approximation of the original problem (Ciarlet[6] and Verhulst[7]). Asymptotic analysis relies mathematically (and particularly considering the open problem of convergence of the asymptotic development for a finite value of ε) on the general techniques developed by Lions[8] for handling linear variational problems containing a small parameter. Methods of asymptotic expansion have been shown to provide a powerful and systematic (although rather formal) tool for justifying two-dimensional plate theories, in both the linear and non-linear cases: the leading term of the asymptotic development of the three-dimensional solution indeed solves the classical equations of the plate theories, with fewer assumptions and, therefore, greater understanding and confidence. Compared with traditional plate theory, the asymptotic method provides more information about the general three-dimensional solution, since it gives additional higher order terms and boundary layer terms.

Asymptotic methods were first applied to plate problems posed as partial differential equations; in that case, some *a priori* assumptions are still needed. For instance, Goldenveizer[9] assumes that the effect of volume forces can be neglected and that the required state of strain and stress is skew-symmetrical about the middle-plane. In addition, some difficulties arise concerning the kind of boundary conditions to be considered for the successive terms of the asymptotic expansion (Friedrichs[10]). A further source of difficulty lies in the absence of a satisfactory convergence analysis, due to both the setting-up of the problem as a set of differential equations (instead of a single one) and the lack of a maximum principle (Eckhaus[11]).

More recently, Ciarlet[12] applied asymptotics to three-dimensional linear plate problems posed in a mixed variational form (displacement-stress approach) called the Hellinger-Reissner variational principle (Washizu[13] and Valid[14]). In that approach, both the displacement and the stresses are considered as unknowns, and such a setting has been shown by previous authors to be the natural one for proving convergence of the development, and for obtaining error estimates.

Quite recently, Klarbring[15] developed an asymptotic model of adhesively-bonded joints on the same basis, where several equivalent variational formulations of the model are given. Of particular interest is the discussion concerning the occurrence and nature of a boundary layer, and the comparison with plate models in terms of deformation modes.

2.2 Contact Between a Rigid Cylindrical Punch and a Thin Elastic Layer: Calculation of the Equivalent Stiffness

The contact law for a thin elastic adhesive has been derived by Klarbring;[15] the first-order solution (superscript 0) corresponds to the situation of an adhesive having a vanishing thickness and is characterised in the following way. The in-plane stresses are given by:

$$\sigma_{12}^0 = 0; \quad \sigma_{11}^0 = \frac{\nu}{1-\nu}\sigma_{33}^0; \quad \sigma_{22}^0 = \frac{\nu}{1-\nu}\sigma_{33}^0 \tag{1}$$

and the normal traction satisfies:

$$\sigma_{i3,3}^0 = 0. \tag{2}$$

$$\sigma_{33}^0 = u_{3,3}^0 \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}. \tag{3}$$

Equation (2) and the expression of the in-plane stress components show that the stress tensor σ^0 is constant through the adhesive thickness, while (3) implies that the displacement u^0 varies linearly through the adhesive thickness, and we can write therefore

$$u^0 = \frac{y_3}{2}(\gamma_1 u^0 - \gamma_2 u^0) + \frac{1}{2}(\gamma_1 u^0 + \gamma_2 u^0) \tag{4}$$

where y_3 is the thickness coordinate and $\gamma_A u^0$ the trace of u^0 on each interfaces S_A, $A = 1, 2$.

These results show that the solution of the first order problem of the expansion does not involve any dependence of the field variables on the thickness coordinate; therefore, the adhesive can be treated as a material surface, letting the mechanical fields within the adhesive depend only on their boundary value on it. Klarbring showed that this situation prevails also for the higher order problems, and the determination of the complete expansion of the displacement and stress fields involves the recursive solution of a sequence of problems, in each of which the adhesive can be considered as a material surface.

We now consider the contact under compression between a flat rigid cylindrical punch and a soft elastic layer of an elastomeric material (for instance, rubber).

It is asumed that there is continuity of displacement at both contact surfaces (perfect adhesion condition), and that the punch is rigid enough (compared with the elastic layer) so that any bending is prevented. From relation (3), we then deduce the expression of the normal stress for a layer having thickness, t (Fig. 1):

$$\sigma_{33}^0 = \frac{U}{t}\frac{E(1-v)}{(1+v)(1-2v)} \tag{5}$$

where v is the Poisson's ratio of the layer and U is the displacement field at the punch-elastic layer boundary, supposed to be under compression. The previous relation is now integrated on the contact surface, resulting in the compression force exerted by the rigid punch:

$$F = -\int_0^a 2\pi r \sigma_{33}^0(r)dr = \pi a^2 U E(1-v)/t(1+v)(1-2v) \tag{6}$$

since the imposed displacement is constant on the contact area, due to the flatness of the punch surface. An equivalent stiffness is defined according to

$$E^e = \frac{Ft}{\pi a^2 U} \tag{7}$$

and we obtain:

$$E^e = \frac{E(1-v)}{(1+v)(1-2v)} \tag{8}$$

the product of the tensile modulus of the layer by a rational function of its Poisson's ratio. This function is nothing else than the compressibility modulus in uniaxial tension, $K^u = \lambda + 2\mu$ (where λ, μ are Lame's coefficients), and we obtain, therefore, as expected

$$E^e = K^u = \lambda + 2\mu. \tag{9}$$

We note that the same result will be obtained whatever the geometrical nature of the area of the punch, provided it remains flat under compression. The obtained value depends only on the layer compressibility; it is seen particularly that the force required

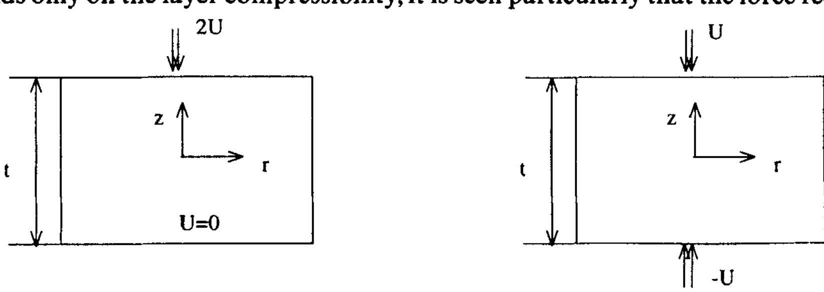

FIGURE 1 Equivalent description of the rigid punch problem.

to compress a very thin layer will increase drastically (and tend to infinity) when the layer tends to incompressibility. This seems physically reasonable since decreasing the thickness of the layer leads to a restriction of its ease of deformation within the bulk, which is prevented from any lateral movement at both contact surfaces.

We are now interested in the mechanical behaviour of the assembly-in terms of the equivalent stiffness – when the thickness of the elastic layer has a finite value. One possible way to obtain the influence of the thickness is to continue the resolution of previous asymptotic expansion towards higher order terms. An examination of the second order solution (Klarbring[15])

$$\sigma^1_{33} = -\sigma^0_{3\alpha,\alpha} y_3 + \hat{\sigma}^1_{33} \tag{10}$$

where $\hat{\sigma}^1_{33}$ is an interfacial stress.

$$u^1_3 = \frac{(1+y_3)}{2}\gamma_1 u^1_3 - \frac{t(1+\nu)(1-2\nu)}{E(1-\nu)}\sigma^0_{3\beta,\beta}\frac{(y^2_3-1)}{2}, \quad \text{with } \gamma_2 u_2 = 0 \tag{11}$$

reveals that the second-order solution involves the first-order stress distribution along the interface, which is difficult to evaluate analytically; therefore we follow another way and try to find first an approximate solution for the displacement.

3. ANALYTICAL SOLUTION OF THE CONTACT PROBLEM

3.1 Determination of an Axisymmetrical Approximate Solution

In order to simplify further the setting of the problem, we consider an equivalent description (Fig. 1): the influence of the material lying outside the contact area is neglected, and the elastic layer is compressed on both sides (normal displacement imposed on both upper and lower faces), the displacement being now half of that of the original problem, in order to keep the same relative displacement of the elastic layer faces. This makes the model symmetrical with respect to the loading.

In the same way as before, it is supposed that perfect adhesion conditions prevail on both interfaces, so that the displacement is continuous. Since both adherends are assumed rigid, this condition implies the nullity of the radial displacement on both faces. Due to the symmetries of the problem, we can reduce the analysis to one quarter of the original domain, *i.e.* to the volume $(r, z) \in [0, a] \times [0, t/2]$ in cylindrical coordinates.

We suppose that each meridian section of the domain deforms by keeping in the same meridian plane, so that we can assume that the angular displacement is zero and that there is no dependence of the field variables on the angular variable. The governing equations of the problem are then deduced from general Lame and Clapeyron conditions (Duc[16]) which are expressed in terms of two differential equations in the radial and normal displacements:

$$(\nabla^2 - 1/r^2)^2 u_r = 0. \tag{12}$$

$$\nabla^4 u_z = 0, \tag{13}$$

when volume forces can be neglected, and where ∇^2 is the Laplacian operator $\nabla^2 = \partial_{rr} + (1/r)\partial_r + (1/r^2)\partial_{\theta\theta}$

The kinematic boundary conditions are:

$$z=t/2:u_z=-U \quad \text{and} \quad u_r=0; \quad z=-t/2:u_z=U \quad \text{and} \quad u_r=0; \quad z=0:u_z=0;$$
$$r=0:u_r=0 \tag{14}$$

and the static conditions are those of null traction on the lateral surface of the layer ($r=a$).

We further simplify the problem by assuming in the case of a thin layer that a section z constant will remain plane, and therefore $\partial u_z/\partial r=0$, which implies that the normal displacement depends only on the normal variable. Under this condition, the biharmonic function f is determined from

$$\nabla^4 f=0 \tag{15}$$

as deduced from (13). We look for a polynomial dependence of f in the variable z, so that the set of boundary conditions to be satisfied implies that the normal displacement is

$$u_z=z\frac{(t/2)^2-z^2}{(t/2)^2}-\frac{Uz}{t/2}. \tag{16}$$

The form of the radial displacement is found by supposing temporarily the material incompressible, which implies the following kinematic condition

$$\frac{\partial u_z}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(ru_r)=0, \tag{17}$$

so that

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r)=2\frac{U}{t}+1-\frac{3z^2}{(t/2)^2} \tag{18}$$

and we deduce that u_r is the product of r by a second degree polynomial in z. The boundary conditions are then satisfied only when

$$u_r=\frac{r(t/2)^2-z^2}{(t/2)^2}. \tag{19}$$

The expression of the radial displacement will be assumed in the following for a rubber layer, which is a quasi-incompressible material.

3.2 Calculation of the Equivalent Stiffness from the Ritz Method

We are looking for an approximate displacement solution of the problem depicted in Figure 1; for that purpose, we first solve the same problem but with homogeneous boundary conditions. Indeed, the space of admissible displacements $U_{ad}=\{(v_r,v_z)\in H^1(\Omega)^2/v_r=0$ at $r=0$; $v_z=0$ at $z=0$; $v_r=0$ and $v_z=-U$ at $z=t/2$; $v_z=U$ and $v_r=0$ at $z=-t/2\}$ becomes a vectorial space under following translation of the solution:

$$u=u_0+\hat{u}, \quad u_0=\left(0,\frac{-U\cdot z}{t/2}\right) \tag{20}$$

and the new displacement, $\hat{u}$, belongs to the translated space, V, with homogeneous boundary conditions:

$$\hat{u}\in V=\{(v_r,v_z)\in H^1(\Omega)^2/v_r=0 \quad \text{at } r=0; \quad v_z=0 \quad \text{at } z=0; \quad v_r=0=v_z \quad \text{at } z=\pm t/2\}.$$

We, therefore, seek an approximate solution for the new variable $\hat{u}$ in the two-dimensional vector space spanned by the vectors $e_1=(r(b^2-z^2/b^2),0)$; $e_2=(0,z\ (b^2-z^2/b^2)$, corresponding to the displacement field given by Eqs. (16), (19), so that we write $\hat{u}(r,z)=y_1e_1+y_2e_2$, where coefficients (y_1,y_2) are determined from a variational formulation described in Appendix 1. The following expression is found:

$$\hat{u}=-3\frac{(\lambda+2\mu)}{\lambda}\left\{e_1+\frac{20U\lambda^2/b}{8\lambda^2-3(\lambda+2\mu)(5\mu a^2/b^2+16(\lambda+\mu))}e^2\right\} \tag{21}$$

and the solution of the original problem (non-homogeneous boundary conditions) is given by (20).

The constitutive law implies the following expression for the normal stress exerted on the upper face of the layer:

$$\sigma_{zz}(r,t/2)=-(\lambda/2\mu)(2y_2+U/b) \tag{21}$$

from which we deduce the total compression force

$$F=(\lambda+2\mu)\pi a^2(2y_2+U/b). \tag{22}$$

We now come back to the original punch problem in which the lower surface of the elastic layer is clamped, so that we multiply the imposed displacement by 2 (*cf*. Fig. 1), resulting in an equivalent stiffness

$$E^e=(\lambda+2\mu)\cdot\left(1-\frac{20\lambda^2}{3(\lambda+2\mu)(5\mu\beta^2+16(\lambda+\mu))-8\lambda^2}\right) \tag{23}$$

in which we recognise the uniaxial compression stiffness under uniaxial tension, $K^u=\lambda+2\mu$. Expression (23) can be simplified after having set

$$\varepsilon=\frac{\mu}{\lambda}; \quad \beta=\frac{a}{t}; \tag{24}$$

$$\frac{E^e}{K^u}=f(\varepsilon,\beta)=1-\frac{20}{40+48(2\varepsilon^2+3\varepsilon)+15\beta^2(2\varepsilon^2+\varepsilon)}. \tag{25}$$

The normalised equivalent stiffness appears, therefore, as a function of the two non-dimensional parameters ε and β.

Parameter ε depends only on Poisson's ratio through $\varepsilon=(1-2\nu/2\nu)$, and tends to zero when the behaviour tends to incompressibility; large (resp. small) values of parameter β (resp. parameter $\gamma=t/a$) are obtained for thin layers (compared with the punch radius).

When the layer becomes very thin, we note that the equivalent stiffness tends to its exact value, K^u, and we should, therefore, expect that the obtained expression gives an accurate effect of the layer thickness, for thin layers.

4. ASSESSMENT OF THE MODEL

4.1 Comparison with Finite Element Calculations

In order to assess the range of validity of the derived model, a numerical simulation using finite elements (FE code TEXPAC[20] used for the calculations is specific for treating nearly incompressible elastic materials like rubber) has been established. The elastic layer is chosen much longer (typically, its length was imposed to be three times the punch radius) than the punch diameter. Both the punch and the elastic layer have been discretized using rectangular quadratic elements (so that the edges of the elements were allowed to be curved), and bending of the contact surface between the layer and the punch was avoided by imposing very high mechanical properties to the punch (this is particularly important when considering a very thin, nearly incompressible layer); the lower face of the layer was prevented from any displacement, while a compressive effort was applied to the upper face of the punch. The calculation then provides the resulting displacement of the contact surface, which was checked to be uniform. Calculations were performed over a range of values of the ratio $\beta \in [1, 100]$ and for a value of Poisson's ratio equal to 0.48, representing a nearly incompressible layer (Young's modulus was taken arbitrarily as 1 MPa, which is a typical value for rubber). For that value, function $f(\varepsilon, \beta)$ expresses as

$$f(\varepsilon, \beta) \approx 1 - \frac{20}{(45,91 + 0,65\beta^2)}. \tag{26}$$

The equivalent normalised stiffness derived from the present model is compared with that obtained from the simulation in Table I and on Figure 2, in which we determine for each value of parameter β the relative error between the simulation and the theory.

It is seen from these results that the theoretical estimate of the equivalent stiffness gives an accurate estimation of the calculated value when the layer thickness is less than one-fourth the punch radius, and can therefore be considered as representative of the actual behaviour of a thin elastic layer being compressed by a flat rigid cylindrical punch. When the ratio a/t becomes very large, numerical calculations underestimate the equivalent stiffness, whereas the analytical expression gives an exact asymptotic value of it. As expected, the theory gives a poor agreement with FE calculations when

TABLE I
Theoretical and calculated equivalent stiffness as a function of geometry ratio

$\beta = a/t$	1	2	4	6	8	10	12	14	16	18
(E^e/K^u) simulation	0.34	0.44	0.63	0.73	0.79	0.83	0.841	0.86	0.878	0.898
(E^e/K^u) theory	0.57	0.587	0.64	0.71	0.77	0.819	0.856	0.884	0.905	0.92
Relative error %	41	25	1.9	3.2	2.4	1.5	1.8	2.75	3	2.5

$\beta = a/t$	20	22	24	26	28	30	40	50	100
(E^e/K^u) simulation	0.9025	0.9134	0.921	0.928	0.934	0.934	0.964	0.965	0.982
(E^e/K^u) theory	0.934	0.94	0.95	0.958	0.964	0.97	0.98	0.988	0.997
Relative error %	3.4	3.3	3.3	3.2	3.1	3.5	1.8	2.3	1.5

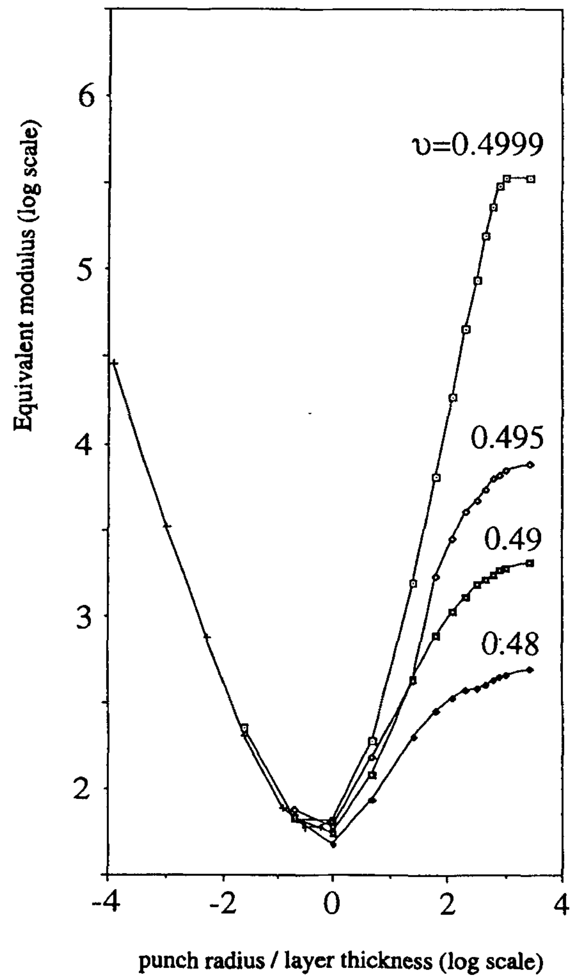

FIGURE 2 Equivalent modulus E^e *vs.* ratio of punch radius, *a*, to layer thickness, *t*, for various values of Poisson's ratio (log scale). *E* denotes the tensile modulus of the elastic layer.

considering geometrical ratios outside this range (when the thickness becomes comparable with, or larger than, the punch radius).

4.2 Effect of Layer Compressibility

The equivalent stiffness (normalised by the traction modulus) has been calculated by finite elements for different values of Poisson's ratio near incompressibility, in the set $\nu \in \{0,48; 0,49; 0,495; 0,49988\}$, Figure 3. For a thin layer, it is shown that even a small variation of the material compressibility has a strong effect on the mechanical behaviour (this is particularly marked when approaching the asymptotic value for very large ratios of a/t). On the other hand, a small effect is observed when the layer thickness becomes comparable with the punch radius; we have checked numerically that there is

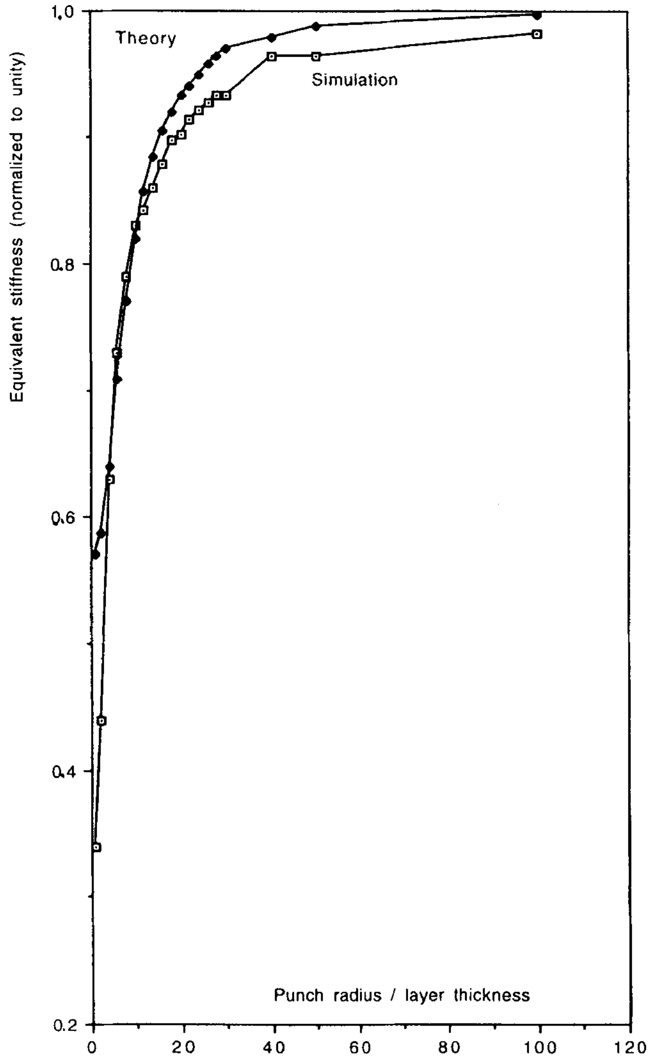

FIGURE 3 Comparison of numerical and analytical evaluations of the equivalent modulus *vs.* ratio of punch radius to layer thickness. Poisson's ratio is equal to 0.48.

no more effect of Poisson's ratio when the thickness is more than ten times as large as the punch radius, the asymptotic value rached for very thick blocks being evidently equal to the Young's modulus of the layer (this can be proved using a variational formulation in term of stresses, resulting in an upper bound of the stiffness, which can be shown to be above the tensile modulus of the material).

In addition, it is seen that the convergence of the equivalent stiffness towards its asymptotic value for thin layers becomes very slow when the behaviour tends to incompressibility, since there is then a drastic increase of the uniaxial compression modulus, which competes with the reduction in thickness necessary for keeping

TABLE II
Effect of Poisson's ratio on convergence of the equivalent stiffness towards its asymptotic value

Poisson's ratio, ν	0.48	0.4825	0.485	0.4875	0.49	0.4925	0.495	0.4975	0.499	0.4995
$\varepsilon = 1 - 2\nu$	0.04	0.035	0.03	0.025	0.02	0.015	0.01	0.005	0.002	0.001
K^u	1284.86	1436.18	1664.64	1984.54	2464.4	3264.3	4864.2	9664.1	24064	48064
(E^e/K^u)	0.92	0.909	0.8966	0.884	0.863	0.837	0.79	0.708	0.56	0.43

normalised stiffness near unity. Such an effect has been assessed numerically, considering a layer thickness twenty times smaller than the punch radius, and making Poisson's ratio vary from 0.48 to 0.4995 (Table II); previously defined parameter ε is approximated by $\varepsilon = 1 - 2\nu$.

It is seen that the normalised stiffness is nearly unity (within a relative error less than 20%) up to a value of Poisson's ratio equal to 0.495; for less compressible materials, the thickness must be much more decreased in order to keep the same global mechanical behaviour: for instance, considering a value of Poisson's ratio equal to 0.4995, a simple calculation shows that the relative variation of the stiffness from its asymptotic value is less than 10% only for a ratio $\beta = a/t$ greater than 100.

A simple calculation shows that the derivative of the normalised stiffness (Eq. 6) for a fixed **finite** value of Poisson's ratio (ε is fixed) has a principal term for a thin layer given by $(\partial f/\partial \gamma) = -8\gamma/3(2\varepsilon^2 + \varepsilon)$. This provides further confirmation of the asymptotic behaviour for thin layers, since this derivative tends to zero with the layer thickness, and the convergence towards the asymptotic value is slower for small values of ε (this parameter appears at the denominator). Since the modelling of very thin layers is a source of numerical troubles, the theoretical model derived previously is used to assess the effect of material compressibility for a thin layer. The equivalent stiffness is developed as an asymptotic expansion *versus* both geometrical and physical small parameters $\gamma = t/a$ and $\varepsilon = 1 - 2\nu$, respectively; this part is presented in Appendix 2.

5. RUPTURE BEHAVIOUR

5.1 Determination of the Rupture Stress from the Solution of the Contact Problem

Adhesion of a rigid flat-ended cylindrical punch to an elastic layer was first treated by Kendall.[21] He evaluated the loss in strain energy, W, in the layer as a circular ring becomes detached at the edge of the flat surface of the punch and spreads inwards. The criterion for rupture propagation for linear elastic systems is obtained from Griffith's fracture criterion

$$-\left(\frac{\partial W}{\partial A}\right)_U \geq G_a \tag{27}$$

where A is the area debonnded ($A = \pi(a^2 - r^2)$ for a punch of radius a with only a central region of radius r still adhering), G_a is the fracture energy per unit of bonded surface and the derivative is taken at constant displacement U.

From the rupture criterion, Eq. (32), the expression for the rupture force, F_a, is easily deduced as

$$F_a^2 = -2G_a/(\partial C/\partial A) = -4\pi r G_a(\partial C/\partial r) \tag{28}$$

where the compliance $C = U/F$ has been introduced.

The average rupture stress, σ_a, can then be expressed as a function of the geometrical parameters r and t using the general expression for the equivalent stiffness, $E^e = (Ft/AU) = g(r,t)K^u$, resulting in

$$\sigma_a^2 = \frac{2E^e G_a}{t} \frac{1}{1 + \dfrac{rg'(r,t)}{2g(r,t)}} \tag{29}$$

where g' denotes $(\partial g/\partial r)$.

Function $g(r,t)$ can be evaluated either by FE calculations, as was done by Ganghoffer,[22] or using the analytical expression derived previously. Using the latter method gives the following expression for the rupture stress:

$$t\sigma_a^2 = 2K^u G_a \frac{f(\varepsilon,\beta)}{1 + \dfrac{3}{4}(2\varepsilon^2 + \varepsilon)\beta^2 \dfrac{(1 - f(\varepsilon,\beta))^2}{f(\varepsilon,\beta)}}. \tag{30}$$

Since the analytical expression for f in (28) is valid with a good accuracy (relative error is less than 3%) when the ratio a/t is greater than 4, the resulting rupture stress also describes correctly the physical reality; the so-determined rupture stress corresponds to the maximum force recorded when exerting traction effort on the punch initially glued on the elastic layer surface.

The relevant rupture variable, $t\sigma_a^2$, is plotted on Figure 4 *versus* the geometrical ratio β, using logarithmic scales for both variables, and material parameters (adhesion energy, traction modulus, Poisson's ratio near 0.5) correspond to the adhesion measurements using a rigid flat punch glued on a soft rubber sheet, rupture occurring

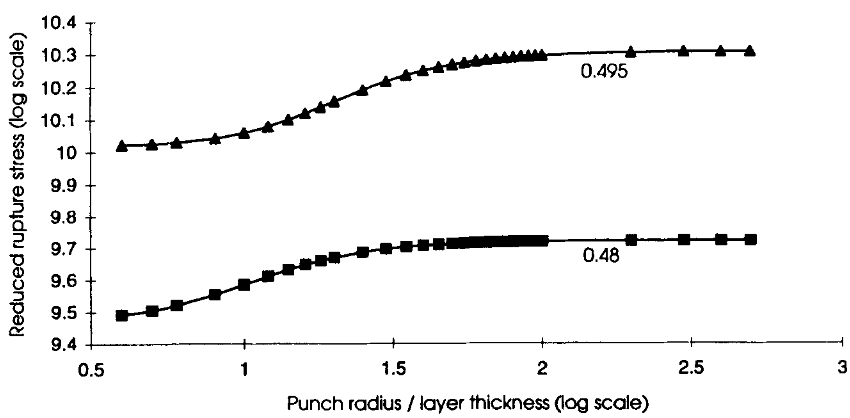

FIGURE 4 Theoretical prediction of rupture stress *vs.* ratio of punch radius to layer thickness. Poisson's ratio is equal to 0.48 and 0.495, as noted.

at the punch/glue interface (Ganghoffer[22]). It is seen that the stress needed for propagating failure increases gradually as the ratio a/t increases, and tends to a limiting value for very thin layers given by $(\sqrt{2K^u G_a}/\sqrt{t})$, since the equivalent modulus tends to a constant value equal to the compressibility modulus K^u (Eq. (29)).

When the ratio a/t is large, compressibility has a strong effect on the effective modulus of the rubber layer and, hence, on the mechanics of fracture. A change of Poisson's ratio from 0.495 to 0.48-which is still representative of rubbery materials-causes a reduction of the rupture force by a factor of about 5, as deduced from (30), Figure 4. This effect points to a need for accurate measurements of compressibility.

5.2 Asymptotic Value of the Rupture Stress for Thin Layers

We consider first the thermodynamics of fracture propagation, following, for instance, the analysis by Maugin.[23] The total dissipation, Φ, can be deduced to be the product of the time derivative of the area created during fracture propagation by its thermodynamic counterpart, G, which is the force due to the singularity at the crack tip: $\Phi = G\dot{A}$. This leads to a crack propagation criterion with a threshold (the surface energy) known as Griffith's criterion: a crack propagates only when G is above a material dependent quantity, G_a. For a rupture that propagates within a material layer (an adhesive, for instance) having thickness, t, in the direction, e_1, the quantity G can be expressed as the contour invariant integral

$$G = \lim_{\Gamma \to 0} \frac{1}{t} \int_{\Gamma} (WN_1 - N_j \sigma_{ij} u_{j,1}) d\Gamma \tag{31}$$

where Γ is a contour around the crack front, W, is the strain energy density and σ is the Cauchy stress tensor. N is the normal to the rupture propagation direction, so that $N_1 = N_i e_{1_i}$ represents the projection of N in the direction of crack propagation. An equivalent expression to (31) is

$$G = \lim_{\rho \to 0} \int_{\Gamma} W dx_2 - t_i \frac{\partial u_i}{\partial x_1} ds. \tag{32}$$

We next express the tensor P on the basis (e_1, N) and a simple calculation leads to following expression:

$$G = \frac{1}{2t} \int_{\Gamma} \sigma_{22} u_{2,2} dx_2. \tag{33}$$

In order to get the asymptotic value of the rupture stress – when the thickness t tends to 0-, we introduce the asymptotic constitutive relation (2) in (12), which results as expected in

$$\sigma_a = \left(\frac{2K^u G_a}{t} \right)^{1/2}. \tag{34}$$

The same expression was obtained by another method by Destuynder,[24] considering a simple kinematics of fracture propagation and that rupture occurs simultaneously within the whole adhesive thickness. It can be seen that the rupture stress tends to infinity when the layer thickness tends to 0. This can be explained as follows: when the thickness of an elastic layer approaches zero, the compliance becomes zero also, which means that no deflection occurs under load. Therefore, no energy is stored to be released by a fracture, and the force required to propagate rupture becomes infinitely large. This is true whatever the mode of deformation of the layer, *i.e.* by dilatation or by shear.

6. CONCLUSION

Analytical models of adhesion problems are scarce in the literature, since the generally complex mechanical behaviour of the adhesive necessitates the use of a numerical method, even for relatively simple geometries. In this paper, an analytical model for the contact and rupture behaviour of an elastic adhesive layer adhered to a rigid punch has been established. An approximate displacement field has been determined, using successively the governing equations for a nearly incompressible material (rubber), and a variational formulation of the problem. The global behaviour of the layer under compression evaluated in this way has been shown to describe accurately (it compares well with F.E. calculations) both geometrical (effect of the layer thickness) and mechanical (effect of layer compressibility) effects. The analytical solution bypasses the limitations encountered in the numerical modelling of thin layers, since it describes the asymptotic behaviour (when the layer thickness vanishes) more accurately than the finite element simulation does.

Perfect adhesion conditions at the interfaces punch/layer and layer/foundation have been assumed in this work, so that the present problem should be extended towards consideration of interfacial decohesion.

Asymptotic methods should find successful applications in the analysis of a wide range of contact and adhesion problems, since they lead to the establishment of contact laws for thin material layers. Treatment of the case where the adhesive or one of the adherends is curved (as in the peel test) or for fiber problems (fragmentation and pull-out test in fiber composite materials), considering differential material behaviour laws are such perspectives.

Appendix 1 *Determination of an approximate displacement field*

Elasticity problems are usually classified according to the nature of the boundary conditions (there are three main classes of problems), and with each is associated a specific appropriate variational formulation; a detailed presentation of variational formulations in linear elasticity can be found in Duvaut.[17]

In the present case, an integral setting equivalent to the differential equations and boundary conditions satisfied by the solution is:

$$\text{(P1): Find} \quad u \in U_{ad} \quad \text{such that} \quad a(u, v-u) = L(v-u), \quad \forall_v \in U_{ad}$$

$$a(u,v) = \int_\Omega (\lambda e(u)e(v) + 2\mu\varepsilon_{ij}(u)\varepsilon_{ij}(v))d\Omega$$

$$L(v) = \int_{\Gamma_1} F_i v_i ds$$

where $e(\cdot)$ is the trace of the small strain tensor $\underline{\underline{\varepsilon}}(\cdot)$, Ω is the volume occupied by the layer, Γ_1 is the portion of the boundary on which tractions F are applied, and U_{ad} is the set of the kinematically admissible fields, that is $U_{ad} = \{v/v_i = u_i \text{ on } \Gamma_0;\ v \in H^1(\Omega)^3\}$, where Γ_0 is the complementary part of the boundary on which displacements are specified.

Considering our particular problem, the space of kinematically admissible fields is

$$U_{ad} = \{(v_r, v_z) \in H^1(\Omega)^2/v_r = 0 \quad \text{at } r=0; \quad v_z = 0 \quad \text{at } z=0; \quad v_r = 0 \quad \text{and}$$

$$v_z = -U \quad \text{at } z=t/2; \quad v_z = U \quad \text{and } v_r = 0 \quad \text{at } z = -t/2\}$$

so that one can identify the boundary Γ_0 with the subset

$$(r,z) \in \{(0,z) \cup (r,t/2), (r,-t/2), (r,0)/r \in [0,a]; z \in [-t/2, t/2]\}$$

and the boundary Γ_1 is the lateral surface of the layer, *i.e.* the set $\{(a,z)/z \in [-t/2, t/2]\}$.

The variational setting of the problem can be shown to be equivalent to the classical differential equations of the linear elasticity problem:

$$\sigma_{ij,i} = 0 \tag{A1.1}$$

$$\sigma_{ij} = A_{ijkl}(x)\varepsilon_{kl}(u) \tag{A1.2}$$

$$u_i = U_i \quad \text{on } \Gamma_0 \tag{A1.3}$$

$$\sigma_{ij} n_j = F_i \quad \text{on } \Gamma_1 \tag{A1.4}$$

Existence and uniqueness of the solution of the variational problem is deduced from application of the Laxmilgram theorem, which relies on mathematical statements concerning the regularity of forms $a(u,v)$, $L(v)$ and coercivity of the bilinear form $a(u,v)$. The solution itselt must have certain regularity properties, and particularly its first-order derivatives must be bounded, so that it belongs to the Sobolev space $H^1 = (\Omega) = \{\varphi/\varphi \in L^2(\Omega);\ (\partial\varphi/\partial x_i) \in L^2(\Omega)\}$, where $L^2(\Omega)$ is the space of squared integrable functions. A detailed analysis of the mathematical statements concerning the functional form of elasticity problems can be found in Duvaut[17,18] and Brezis.[19] Such mathematical considerations are not needed for the understanding of further developments and we, therefore, stay with the elementary presentation given above.

Considering our particular problem, we note that the space of admissible displacements

$$U_{ad} = \{(v_r, v_z) \in H^1(\Omega)^2/v_r = 0 \quad \text{at } r=0; \quad v_z = 0 \quad \text{at } z=0; \quad v_r = 0 \quad \text{and}$$

$$v_z = -U \quad \text{at } z=t/2; \quad v_z = U \quad \text{and } v_r = 0 \quad \text{at } z = -t/2$$

becomes a vectorial space under following translation of the solution: $u = u_0 + \hat{u}$, with $u_0(0, -U \cdot z/t/2)$, and the new displacements, $\hat{u}$, belongs to the translated space, V, with homogeneous boundary conditions:

$$\hat{u} \in V = \{(v_r, v_z) \in H^1(\Omega)^2/v_r = 0 \quad \text{at } r=0; \quad v_z = 0 \quad \text{at } z=0; \quad v_r = 0 = v_z \quad \text{at } z = \pm t/2\}$$

Since no surface forces are applied, the linear form L is null and, according to previous translation of the solution, we obtain the following equivalent problem:

(P2): Find $\hat{u} \in V$ such that

$$a(\hat{u}, v) = -a(u_0, v) = \frac{2U}{t} \int_\Omega (\lambda e(v) + 2\mu \varepsilon_z(v)) \cdot d\Omega$$

where the last integral reduces to $(2U\lambda/t)\int_\Omega e(v) d\Omega$.

We seek the solution of this problem in a finite dimensional space, spanned by the two vectors associated with the approximate displacement field found in the previous paragraph:

$$e_1 = \left(r\frac{b^2 - z^2}{b^2}, 0\right); \quad e_2 = \left(0, z\frac{b^2 - z^2}{b^2}\right)$$

where we have set $b = t/2$. The approximate solution, $\hat{u}$, is written in this two-dimensional space as $\hat{u} = y_1 e_1 + y_2 e_2$, where y_1, y_2 are two constants, and (P2) now becomes:

(P3): Find $(y_1, y_2) \in R^2$ such that

$$a(e_1, e_1) y_1 + a(e_1, e_2) y_2 = -a(u^0, e_1)$$
$$a(e_1, e_2) y_1 + a(e_2, e_2) y_2 = 0.$$

As a matter of simplification, we write a_{ij} instead of $a(e_i, e_j)$ in the following.
A simple calculation gives the term $-a(u^0, e_1) = (4/3)(\pi U \lambda a^2)$.

We first evaluate the strain tensor components (in cylindrical coordinates) associated with each basis vector:

$$\varepsilon_r(e_1) = \varepsilon_\theta(e_1) = 1 - \left(\frac{z}{b}\right)^2; \quad \varepsilon_z(e_1) = 0; \quad \varepsilon_{rz}(e_1) = 0. \quad \text{(A1.5)}$$

$$\varepsilon_r(e_2) = \varepsilon_\theta(e_2) = 0; \quad \varepsilon_z(e_2) = 1 - 3\left(\frac{z}{b}\right)^2; \quad \varepsilon_{rz}(e_2) = 0. \quad \text{(A1.6)}$$

The determination of coefficients a_{ij} follows after elementary calculations:

$$a_{11} = \frac{32}{15}\pi a^2 b(\lambda + \mu) + \frac{2}{3}\pi a^2 b \mu \frac{a^2}{b^2}; \quad a_{22} = \frac{4}{5}\pi a^2 b(\lambda + 2\mu); \quad a_{12} = \frac{8}{15}\pi a^2 b \lambda. \quad \text{(A1.7)}$$

The evaluation of the coefficients y_1, y_2, follows:

$$y_1 = -3\frac{(\lambda + 2\mu)}{\lambda} y_2; \quad y_2 = \frac{20 U \lambda^2 / b}{8\lambda^2 - 3(\lambda + 2\mu)(5\mu a^2/b^2 + 16(\lambda + \mu))}. \quad \text{(A1.8)}$$

Appendix 2 ***Asymptotic expansion of the equivalent stiffness versus layer thickness and compressibility***

We obtain further insight into the influence of a finite value of Poisson's ratio by making a series development of the normalised stiffness (24) in terms of the geometrically small parameter, γ:

$$\frac{E^e}{K^u} = 1 - \frac{4\gamma^2}{3(2\varepsilon^2+\varepsilon)} - 2\gamma^3 \frac{(2+12/5(2\varepsilon^2+3\varepsilon)}{(3/4(2\varepsilon^2+\varepsilon))^{1/2}} + o(\gamma)^3 \quad \text{(A2.1)}$$

Development (A2.1) is an asymptotic expansion in the variable γ, for a finite value of parameter ε. We are now interested in deriving such an expansion when **both** parameters, ε and γ, tend to zero. Since the ratio (γ/ε) intervenes in the derivation of this development, we make the hypothesis that γ is of the form

$$\gamma = \theta\varepsilon^\delta \quad \text{(A2.2)}$$

where $\delta > 1$ and θ are fixed constants.

We justify this model by the fact that the denominator of expression (A2.1) of the normalised stiffness-when written in variables γ and ε-contains powers of ε/γ; since the normalised stiffness tends to unity when the layer thickness tends to zero $(\gamma \to 0)$ whatever the value of ε (thus, also, in the case when both parameters tend to zero), this implies the value $\delta > 1$.

Introducing relation (A2.2) into Eq. (A2.1) leads to a double asymptotic expansion of the normalised stiffness given by

$$\frac{E^e}{K^u} = 1 - \frac{4}{3}\frac{\gamma}{\varepsilon} - \frac{8}{3}\gamma - 2\left(\frac{4}{3}\right)^2\frac{\gamma^2}{\varepsilon} - \frac{12}{5}\left(\frac{4}{3}\right)^2\gamma^2 - \frac{12}{5}\left(\frac{4}{3}\right)^2\gamma^2\varepsilon + o(\gamma^2\varepsilon) \quad \text{(A2.3)}$$

and the constraint $\delta > 1$ implies that the ratio γ/ε tends to zero with γ.

This development (which evidently depends on the analytical model built) makes clear that the mechanical behaviour of the elastic layer under contact with the punch is influenced by the mutual interplay of geometrical and physical small parameters, $\gamma = t/a$ and $\varepsilon \approx 1-2\nu$, respectively.

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